

A Riemann-Hilbert approach to the Harry-Dym equation on the line

Yu Xiao¹, Engui Fan^{1*}

¹ School of Mathematical Science, Fudan University, Shanghai 200433, P.R. China

Abstract

In this paper, we consider the Harry-Dym equation on the line with decaying initial value. The Fokas unified method is used to construct the solution of the Harry-Dym equation via a 2×2 matrix Riemann Hilbert problem in the complex plane. Further, one-cups solution is expressed in terms of solutions of the Riemann Hilbert problem.

Keywords: Harry-Dym equation, Riemann-Hilbert problem, Initial-value problem, One-cups solution

1 Introduction

The following nonlinear partial differential equation

$$q_t - 2\left(\frac{1}{\sqrt{1+q}}\right)_{xxx} = 0 \quad (1.1)$$

is known as the Harry-Dym equation [1]. This equation was obtained by Harry-Dym and Martin Kruskal as an evolution equation solvable by a spectral problem based on the string equation instead of the Schrodinger equation. The Harry-Dym equation has interest in the study of the Saffman-Taylor problem which describes the motion of a two-dimensional interface between a viscous and a nonviscous fluid [2]. The Harry-Dym equation

*Corresponding author and e-mail address: faneg@fudan.edu.cn

shares many of the properties typical of the soliton equations. It is a completely integrable equation which can be solved by the inverse scattering transform [3]. It has a bi-Hamiltonian structure [4], an infinite number of conservation laws and infinitely many symmetries [5], and has reciprocal Backlund transformations to the KdV equation [6]. The Harry-Dym equation has been solved in different method such as the inversing scattering method [3], the Bäcklund transformation technique [7], the straightforward method [8]. Especially, the Wadati obtained the one-cups soliton solution [3]

$$q(x, t) = \tanh^{-4}(\kappa x - 4\kappa^3 t + \kappa x_0 + \varepsilon_+) - 1,$$

$$\varepsilon_+ = \frac{1}{\kappa}[1 + \tanh(\kappa x - 4\kappa^3 t + \kappa x_0 + \varepsilon_+)].$$

by using inverse scattering transformation.

The main aim of this paper is to develop the inversing scattering method, based on an Riemann-Hilbert problem for solving nonlinear integrable systems called unified method [9] which has been further developed and applied in different equations with initial value problems on the line [10, 11, 12, 13, 14] and initial boundary value problem on half line [15, 16, 17]. In this paper, we consider the initial value problem of the Harry-Dym equation

$$q_t - 2\left(\frac{1}{\sqrt{1+q}}\right)_{xxx} = 0, \quad x \in R, \quad t > 0, \quad (1.2)$$

$$q(x, 0) = q_0(x),$$

where the $q_0(x)$ is a smoothly real-valued function and decay as $|x| \rightarrow \infty$. The organization of the paper is as follows. In the following section 2, we perform the spectral analysis of the associated Lax pair for the Harry-Dym equation. In section 3, we formulate the main Riemann-Hilbert problem associated with the initial value problem (1.2). In section 4, we obtain one-cups soliton solution in the terms of Riemann-Hilbert problem, which has a similar, but not the same form constructed by the inverse scattering method [3].

2 Spectral analysis

2.1 A Lax pair

In general, the matrix Riemann Hilbert problem is defined in the λ plane and has explicit (x, t) dependence, while for the Harry-Dym equation (1.2), we need to construct a new matrix Riemann Hilbert problem with explicit (y, t) dependence, where $y(x, t)$ is a function which is an unknown from the initial value condition. For this purpose, we make a transformation

$$\rho = \sqrt{1 + q},$$

the equation (1.2) can be expressed by

$$(\rho^2)_t - 2\left(\frac{1}{\rho}\right)_{xxx} = 0.$$

Then the initial value problem (1.2) is transformed into

$$(\rho^2)_t - 2\left(\frac{1}{\rho}\right)_{xxx} = 0, x \in R, t > 0, \quad (2.1)$$

$$\rho(x, 0) = \rho_0(x) = \sqrt{1 + q_0(x)},$$

$$\rho_0(x) \rightarrow 1, |x| \rightarrow \infty.$$

It was shown that the equation (1.2) admits the following Lax pair [3]

$$\begin{cases} \psi_{xx} = -\lambda^2(1 + q)\psi, \\ \psi_t = 2\lambda^2\left[\frac{2}{\sqrt{1+q}}\psi_x - \left(\frac{1}{\sqrt{1+q}}\right)_x\psi\right]. \end{cases} \quad (2.2)$$

Making a transformation

$$\rho = \sqrt{1 + q}, \quad \varphi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix},$$

then the Lax pair (2.2) can be written in matrix form

$$\begin{cases} \varphi_x = M\varphi, \\ \varphi_t = N\varphi, \end{cases} \quad (2.3)$$

where

$$M = \begin{pmatrix} 0 & 1 \\ -\lambda^2 \rho^2 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -2\lambda^2(\frac{1}{\rho})_x & 4\lambda^2 \frac{1}{\rho} \\ -4\lambda^4 \rho - 2\lambda^2(\frac{1}{\rho})_{xx} & 2\lambda^2(\frac{1}{\rho})_x \end{pmatrix}.$$

Further by the gauge transformations

$$\phi = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda \rho} & 0 \\ 0 & \frac{1}{\sqrt{\lambda \rho}} \end{pmatrix} \varphi.$$

we have

$$\begin{cases} \phi_x + i\lambda\rho\sigma_3\phi = U\phi, \\ \phi_t + i(\lambda\frac{1}{\rho}(\frac{1}{\rho})_{xx} + 4\lambda^3)\sigma_3\phi = V\phi, \end{cases} \quad (2.4)$$

where

$$U(x, t) = \frac{1}{2} \frac{\rho_x}{\rho} \sigma_2, \quad V(x, t, \lambda) = -\lambda \frac{1}{\rho} (\frac{1}{\rho})_{xx} \sigma_1 - 2\lambda^2 (\frac{1}{\rho})_x \sigma_2.$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is clear that as $|x| \rightarrow \infty$, $U(x, t) \rightarrow 0$, $V(x, t, \lambda) \rightarrow 0$. We define a real-valued function $y(x, t)$ by

$$y(x, t) = x + \int_x^\infty (1 - \rho(\xi, t)) d\xi.$$

It is obvious that

$$y_x = \rho(x, t), \quad y_t = - \int_x^\infty \rho_t(\xi, t) d\xi.$$

The conservation law

$$\rho_t - (-\frac{1}{2}((\frac{1}{\rho})_x)^2 + \frac{1}{\rho}(\frac{1}{\rho})_{xx})_x = 0$$

implies that

$$y_t = -\frac{1}{2}((\frac{1}{\rho})_x)^2 + \frac{1}{\rho}(\frac{1}{\rho})_{xx}.$$

Extending the column vector ϕ to be a 2×2 matrix and letting

$$\mu = \phi \exp(i\lambda y(x, t)\sigma_3 + 4i\lambda^3 t\sigma_3),$$

then μ solves

$$\begin{cases} \mu_x + i\lambda y_x[\sigma_3, \mu] = \tilde{U}\mu, \\ \mu_t + i(\lambda y_t + 4\lambda^3)[\sigma_3, \mu] = \tilde{V}\mu, \end{cases} \quad (2.5)$$

which can be written in full derivative form

$$d(e^{i(y(x,t)x+4\lambda^3t)\hat{\sigma}_3}\mu) = e^{i(y(x,t)x+4\lambda^3t)\hat{\sigma}_3}(\tilde{U}dx + \tilde{V}dt)\mu,$$

where

$$\begin{aligned} \tilde{U} &= U, \\ \tilde{V} &= -\frac{1}{2}i\lambda\left(\left(\frac{1}{\rho}\right)_x\right)^2\sigma_3 - \lambda\frac{1}{\rho}\left(\frac{1}{\rho}\right)_{xx}\sigma_1 - 2\lambda^2\left(\frac{1}{\rho}\right)_x\sigma_2, \end{aligned}$$

$[\sigma_3, \mu] = \sigma_3\mu - \mu\sigma_3$. As $|x| \rightarrow \infty$, $\tilde{V} \rightarrow 0$. The lax pair in the (2.5) is very convenient for dedicated solutions via integral Volterra equation, which is also what we study in the following paper.

Remark 2.1 By the representation of M, N and U, V in (2.3) and (2.4) respectively, we find that ψ_x, ψ_t and ϕ_x, ϕ_t have no singularity in $\lambda = 0$. Therefore, ϕ has no real singularity in $\lambda = 0$.

2.2 Eigenfunctions We define two eigenfunctions μ_{\pm} of equation (2.5) as the solutions of the following two Volterra integral equation in the (x, t) plane

$$\mu(x, t, \lambda) = I + \int_{(x^*, t^*)}^{(x, t)} e^{-[i\lambda(y(x, t) - y(x', t)) + 4i\lambda^3(t - \tau)]\hat{\sigma}_3} (\tilde{U}(x', t)\mu(x', t, \lambda)dx' + \tilde{V}(x', \tau, \lambda)\mu(x', \tau, \lambda))d\tau \quad (2.6)$$

where I is the 2×2 identity matrix, $\hat{\sigma}_3$ acts on a 2×2 matrix A by $\hat{\sigma}_3 A = \sigma_3 A \sigma_3$. Since the integrated expression is independent of the path of integration, we choose the particular initial points of integration to be parallel to the x -axis and obtain that μ_+ and μ_-

$$\mu_+(x, t, \lambda) = I - \int_x^\infty e^{-i\lambda(y(x, t) - y(x', t))\hat{\sigma}_3} \tilde{U}(x', t)\mu_+(x', t, \lambda)dx',$$

$$\mu_-(x, t, \lambda) = I + \int_{-\infty}^x e^{-i\lambda(y(x,t)-y(x',t))\sigma_3} \tilde{U}(x', t) \mu_-(x', t, \lambda) dx'. \quad (2.7)$$

Define the following sets

$$D_1 = \{\lambda \in C | \text{Im}\lambda > 0\},$$

$$D_2 = \{\lambda \in C | \text{Im}\lambda < 0\}.$$

Since any fixed t , $y_x = \rho(x, t) > 0$, $y(x, t)$ is increasing function of x for fixed t . as $x - x' < 0$, $y(x, t) - y(x', t) < 0$; as $x - x' > 0$, $y(x, t) - y(x', t) > 0$. We can deduce that the second column vectors of μ_+ , μ_- are bounded and analytic for $\lambda \in C$ provided that λ belongs to D_1, D_2 , respectively. We denote these vectors with superscripts (1),(2) to indicate the domains of their boundedness. Then

$$\mu_+ = (\mu_+^{(2)}, \mu_+^{(1)}), \mu_- = (\mu_-^{(1)}, \mu_-^{(2)}).$$

For any x, t , the following conditions are satisfied

$$(\mu_-^{(1)}, \mu_+^{(1)}) = I + O(1/\lambda), \lambda \rightarrow \infty, \lambda \in D_1,$$

$$(\mu_+^{(2)}, \mu_-^{(2)}) = I + O(1/\lambda), \lambda \rightarrow \infty, \lambda \in D_2,$$

$$\mu_{\pm} = I + O(1/\lambda), \lambda \rightarrow \infty.$$

2.3 Spectral functions For $\lambda \in R$, the eigenfunction μ_+, μ_- being the solution of the system of differential equation (2.5) are related by a matrix independent of (x, t) . We define the spectral function by

$$\mu_+(x, t, \lambda) = \mu_-(x, t, \lambda) e^{-i(\lambda y(x,t) + 4\lambda^3 t)\sigma_3} s(\lambda). \quad (2.8)$$

From (2.5), we get

$$\det(\mu_{\pm}(x, t, \lambda)) = 1. \quad (2.9)$$

Since $\overline{\tilde{U}}(x, t) = -\tilde{U}(x, t)$, the $\mu_{\pm}(x, t, \lambda)$ have the relations

$$\begin{cases} \mu_{\pm 11}(x, t, \lambda) = \overline{\mu_{\pm 22}(x, t, \bar{\lambda})}, & \mu_{\pm 21}(x, t, \lambda) = \overline{\mu_{\pm 12}(x, t, \bar{\lambda})}, \\ \mu_{\pm 11}(x, t, -\lambda) = \mu_{\pm 22}(x, t, \lambda), & \mu_{\pm 12}(x, t, -\lambda) = \mu_{\pm 21}(x, t, \lambda). \end{cases} \quad (2.10)$$

The spectral function $s(\lambda)$ can be written as

$$s(\lambda) = \begin{pmatrix} \overline{a(\bar{\lambda})} & b(\lambda) \\ \overline{b(\bar{\lambda})} & a(\lambda) \end{pmatrix}, \quad (2.11)$$

$$s(\lambda) = I - \int_{-\infty}^{+\infty} e^{i\lambda y(x,0)\hat{\sigma}_3} \tilde{U}(x', 0) \mu_+(x', 0, \lambda) dx', \quad \text{Im}\lambda = 0. \quad (2.12)$$

From the (2.9), $\det(s(\lambda)) = 1$. Equation (2.8) and (2.9) imply $a(\lambda)$ and $b(\lambda)$ have the following properties:

- $a(\lambda)$ is analytic in D_1 and continuous for $\lambda \in \bar{D}_1$.
- $b(\lambda)$ is continuous for $\lambda \in R$.
- $a(\lambda)\overline{a(\bar{\lambda})} - b(\lambda)\overline{b(\bar{\lambda})} = 1, \quad \lambda \in R$.
- $a(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty, \quad \lambda \in D_1$.
- $b(\lambda) = O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty, \quad \lambda \in R$.

2.4 Residue conditions We assume that $a(\lambda)$ has N simple zeros $\{\lambda_j\}_{j=1}^N$ in the upper half plane. These eigenvalues are purely imaginary. The second column of equation (2.8) is

$$\mu_+^{(1)} = b(\lambda)\mu_-^{(1)} e^{-2i(\lambda y(x,t)+4\lambda^3 t)} + \mu_-^{(2)} a(\lambda). \quad (2.13)$$

For the (2.9) and equation (2.13), it yields

$$a(\lambda) = \det(\mu_-^{(1)}, \mu_+^{(1)})$$

where we have used that both sides are well defined and analytic in D_1 to extend the above relation to \bar{D}_1 . Hence if $a(\lambda_j) = 0$, the $\mu_-^{(1)}, \mu_+^{(1)}$ are linearly dependent vectors for each x and t , i.e. there exist constants $b_j \neq 0$ such that

$$\mu_-^{(1)} = b_j e^{2i(\lambda_j y(x,t)+4\lambda_j^3 t)} \mu_+^{(1)}, \quad x \in R, t > 0.$$

Recalling the symmetries in the (2.10), we find

$$\mu_-^{(2)} = \bar{b}_j e^{-2i(\bar{\lambda}_j y(x,t) + 4\bar{\lambda}_j^3 t)} \mu_+^{(2)}, x \in R, t > 0.$$

Consequently, the residues of $\mu_-^{(1)}/a$ and $\mu_-^{(2)}/\overline{a(\bar{\lambda})}$ at λ_j and $\bar{\lambda}_j$ are

$$\begin{aligned} \text{Res}_{\lambda=\lambda_j} \frac{\mu_-^{(1)}(x, t, \lambda)}{a(\lambda)} &= C_j e^{2i(\lambda_j y(x,t) + 4\lambda_j^3 t)} \mu_+^{(2)}(x, t, \lambda_j), \quad j = 1, \dots, N, \\ \text{Res}_{k=\bar{\lambda}_j} \frac{\mu_-^{(2)}(x, t, \lambda)}{\overline{a(\lambda)}} &= \bar{C}_j e^{-2i(\bar{\lambda}_j y(x,t) + 4\bar{\lambda}_j^3 t)} \mu_+^{(1)}(x, t, \bar{\lambda}_j), \quad j = 1, \dots, N, \end{aligned}$$

where $C_j = \frac{b_j}{\dot{a}(k_j)}$, $\dot{a}(k) = \frac{da}{dk}$.

Remark 2.2 There is the relation of μ_{\pm} that the $s(\lambda)$ is the scattering matrix for the one dimensional Schrödinger equation

$$W_{yy} + \lambda^2 W = f(y)W$$

via the Liouville transformation:

$$y = x + \int_x^\infty (1 - \rho(\xi, 0)) d\xi, \quad W(y, \lambda) = \psi(y, \lambda) \rho_0(y)$$

$$\rho_0(y) = \rho_0(x), \quad f(y) = \frac{1}{2}(\rho_{0yy} \rho_0^{-1} - \frac{1}{2} \rho_{0y}^2 \rho_0^{-2}).$$

Therefore, in terms of spectral problem of Schrödinger equation, we deduce that $a(\lambda)$ only has pure imaginary of simple poles in the upper plane.

3 The Riemann-Hilbert Problem

3.1 A Riemann-Hilbert problem for (x, t) We now apply uniform method to solve the initial value problem for equation (2.1) on the line, and the solution can be expressed in terms of a 2×2 matrix Riemann-Hilbert problem. Let $M(x, t, \lambda)$ be defined by

$$M_+ = \left(\frac{\mu_-^{(1)}}{a(\lambda)}, \mu_+^{(1)} \right), \quad \lambda \in D_1; \quad M_- = \left(\mu_+^{(2)}, \frac{\mu_-^{(2)}}{a(\bar{\lambda})} \right), \quad \lambda \in D_2 \quad (3.1)$$

and the M satisfied the jump condition:

$$M_+(x, t, \lambda) = M_-(x, t, \lambda)J(x, t, \lambda), \quad \text{Im}\lambda = 0,$$

where

$$J(x, t, \lambda) = \begin{pmatrix} \frac{1}{a(\lambda)\overline{a(\lambda)}} & \frac{b(\lambda)}{a(\lambda)}e^{-2i(\lambda y(x,t)+4\lambda^3 t)} \\ -\frac{\overline{b(\lambda)}}{a(\lambda)}e^{2i(\lambda y(x,t)+4\lambda^3 t)} & 1 \end{pmatrix}, \quad \text{Im}\lambda = 0. \quad (3.2)$$

These definitions imply

$$\det M(x, t, \lambda) = 1 \quad (3.3)$$

and

$$M(x, t, \lambda) = I + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty. \quad (3.4)$$

This contour of RH problem is the real axis.

the jump matrix $J(x, t, \lambda)$, the spectral $a(\lambda)$ and $b(\lambda)$ are dependent on the $y(x, t)$, while $y(x, t)$ is not involve initial data. Therefore, this RH problem can't be formulated in terms of initial alone. In order to overcome this problem, we will reconstruct a new jump matrix by changing

$$(x, t) \rightarrow (y, t), \quad y = y(x, t),$$

y is a new scale. Then we can transform this RH problem into the RH problem parametrized by (y, t) .

3.2 A Riemann-Hilbert problem for (y, t)

3.1 The theorem Let $q_0(x), x \in R$ be a smooth function and decay as $|x| \rightarrow \infty$. Moreover $1 + q_0(x) > 0$. Define the \tilde{U}_0, ρ_0 and $y_0(x)$ as follows:

$$\tilde{U}_0(x) = \frac{1}{2} \frac{\rho_{0x}(x)}{\rho_0(x)} \sigma_2, \quad \rho_0(x) = \sqrt{1 + q_0(x)},$$

$$y_0(x) = x + \int_x^\infty (1 - \rho_0(\xi)) d\xi.$$

Let $\mu_+(x, 0, \lambda)$ and $\mu_-(x, 0, \lambda)$ be the unique solution of the Volterra linear integral equation (2.5) evaluated at $t = 0$ with $\tilde{U}_0(x, 0) = \tilde{U}_0(x), \rho_0(x) =$

$\rho(x, 0)$ and $y_0(x) = y(x, 0)$. Define $a(\lambda), b(\lambda), C_j$ by

$$\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = [s(\lambda)]_2, \quad s(\lambda) = I - \int_{-\infty}^{+\infty} e^{i\lambda y_0(x)\hat{\sigma}_3} \tilde{U}_0(x') \mu_+(x', 0, \lambda) dx', \quad \text{Im}\lambda = 0 \quad (3.5)$$

and

$$[\mu_-(x, 0, \lambda_j)]_1 = \dot{a}(\lambda_j) C_j e^{2i\lambda_j y_0(x)} [\mu_+(x, 0, \lambda_j)]_2, \quad j = 1, \dots, N, \quad (3.6)$$

where $([A]_1 \ [A]_2)$ denotes the first (second) column of a 2×2 matrix A . We assume that $a(\lambda)$ has N simple zeros $\{\lambda_j\}_{j=1}^N$ in the upper half plane and are pure imaginary. Then

- $a(\lambda)$ is defined for $k \in \bar{D}_1$ and analytic in D_1 .
- $b(\lambda)$ is defined for $\lambda \in R$.
- $a(\lambda)\overline{a(\bar{\lambda})} - b(\lambda)\overline{b(\bar{\lambda})} = 1, \quad \lambda \in R$.
- $a(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \lambda \rightarrow \infty, \lambda \in D_1$.
- $b(\lambda) = O\left(\frac{1}{\lambda}\right), \lambda \rightarrow \infty, \lambda \in R$.

Suppose there exists a uniquely solution $q(x, t)$ of equation (1.2) with initial data $q_0(x)$ such that $\rho_0(x) = \sqrt{1 + q_0(x)}$ has sufficient smoothness and decay for $t > 0$. Then $q(x, t)$ is given in parametric form by

$$q(x(y, t), t) = e^{8 \int_y^{+\infty} m(y', t) dy'} - 1 \quad (3.7)$$

and the function $x(y, t)$ is defined by

$$x(y, t) = y + \int_{-\infty}^y (e^{-4 \int_{y'}^{\infty} m(\xi, t) d\xi} - 1) dy', \quad (3.8)$$

where $m(y, t) = -i \lim_{\lambda \rightarrow \infty} (\lambda M(y, t, \lambda))_{12}$, and $M(y, t, \lambda)$ is the uniquely solution of the following RH problem

- $M(y, t, \lambda) = \begin{cases} M_-(y, t, \lambda), & \lambda \in D_2, \\ M_+(y, t, \lambda), & \lambda \in D_1. \end{cases}$
is a sectionally meromorphic function.

- $M_+(y, t, \lambda) = M_-(y, t, \lambda)J^{(y)}(y, t, \lambda), \quad \text{Im}\lambda = 0,$

where $J^{(y)}(y, t, \lambda)$ is defined by

$$J^{(y)}(y, t, \lambda) = \begin{pmatrix} \frac{1}{a(\lambda)\overline{a(\lambda)}} & \frac{b(\lambda)}{a(\lambda)}e^{-2i(\lambda y + 4\lambda^3 t)} \\ -\frac{\overline{b(\lambda)}}{a(\lambda)}e^{2i(\lambda y + 4\lambda^3 t)} & 1 \end{pmatrix}, \quad \text{Im}\lambda = 0. \quad (3.9)$$

•

$$M(y, t, \lambda) = I + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty. \quad (3.10)$$

- The possible simple poles of the first column of $M_+(y, t, \lambda)$ occur at $\lambda = \lambda_j, j = 1, \dots, N$, and the possible simple poles of the second column of $M_-(y, t, \lambda)$ occur at $\lambda = \bar{\lambda}_j, j = 1, \dots, N$. The associated residues are given by

$$\text{Res}_{\lambda=\lambda_j} [M(y, t, \lambda)]_1 = C_j e^{2i(\lambda_j y + 4\lambda_j^3 t)} [M(y, t, \lambda_j)]_2, \quad j = 1, \dots, N, \quad (3.11)$$

$$\text{Res}_{\lambda=\bar{\lambda}_j} [M(y, t, \lambda)]_2 = \bar{C}_j e^{-2i(\bar{\lambda}_j y + 4\bar{\lambda}_j^3 t)} [M(y, t, \bar{\lambda}_j)]_1, \quad j = 1, \dots, N. \quad (3.12)$$

Proof: Assume that $\mu(x, t)$ is the solution of equation (2.5), the asymptotic expansion of it

$$\mu(x, t, \lambda) = I + \frac{\mu^{(1)}(x, t)}{\lambda} + \frac{\mu^{(2)}(x, t)}{\lambda^2} + \frac{\mu^{(3)}(x, t)}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right), \lambda \rightarrow \infty$$

into the x -part of equation (2.5), where $\mu^{(1)}(x, t), \mu^{(2)}(x, t)$ and $\mu^{(3)}(x, t)$ are 2×2 matrixes, dependent on x, t . by considering the terms of $O(1)$, We get

$$4\mu_{12}^{(1)}(x, t) = -\frac{\rho_x(x, t)}{\rho(x, t)}. \quad (3.13)$$

By construction of the new RH problem about (y, t, λ) , we can deduce that

$$\mu_{12}^{(1)}(x, t) = -i \lim_{\lambda \rightarrow \infty} (\lambda M(y, t, \lambda))_{12} = m(y, t). \quad (3.14)$$

Then

$$-\frac{1}{4} \frac{\rho_x(x, t)}{\rho(x, t)} = m(y, t). \quad (3.15)$$

Equation (3.13) can be expressed in terms of $y = y(x, t)$. Indeed, using $\frac{dy}{dx} = \rho$, then (3.15) becomes

$$-\frac{1}{4} \frac{\rho_y}{\rho} = m(y, t). \quad (3.16)$$

As $|y| \rightarrow \infty$, $\rho(y, t) \rightarrow 1$, by the evaluation of (3.16), we get

$$\rho(y, t) = e^{4 \int_y^{+\infty} m(y', t) dy'}$$

Therefore,

$$q(x, t) = e^{8 \int_y^{+\infty} m(y', t) dy'} - 1$$

As $|x| \rightarrow \infty$, $|y| \rightarrow \infty$ and $\frac{dy}{dx} = \rho > 0$, so

$$x = y + \int_{-\infty}^y (e^{-4 \int_{y'}^{+\infty} m(\xi, t) d\xi} - 1) dy'.$$

Remark 3.1 It follows from the symmetries (2.10) that the solution $M(y, t, \lambda)$ of Riemann Hilbert problem in the 3.1 theorem has the symmetries

$$\begin{cases} M_{11}(y, t, \lambda) = \overline{M_{22}(y, t, \bar{\lambda})}, & M_{21}(y, t, \lambda) = \overline{M_{12}(y, t, \bar{\lambda})}, \\ M_{11}(y, t, -\lambda) = M_{22}(y, t, \lambda), & M_{12}(y, t, -\lambda) = M_{21}(y, t, \lambda). \end{cases} \quad (3.17)$$

4 Soliton solution

The solitons correspond to spectral data $\{a(\lambda), b(\lambda), C_j\}$ for which $b(\lambda)$ vanishes identically. In this case the jump matrix $J^{(y)}(y, t, \lambda)$ in the (3.9) is the identity matrix and the RH problem of 3.1 theorem consists of finding a meromorphic function $M(y, t, \lambda)$ satisfying (3.10) and the residue conditions (3.11) and (3.12). From (3.10) and (3.11), we get

$$[M(y, t, \lambda)]_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^N \frac{C_j}{\lambda - \lambda_j} e^{2i(\lambda_j y + 4\lambda_j^3 t)} [M(y, t, \lambda_j)]_2. \quad (4.1)$$

for the symmetries (3.17), equation (4.1) can be written as

$$\left(\frac{\overline{M_{22}(y, t, \bar{\lambda})}}{M_{12}(y, t, \bar{\lambda})} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^N \frac{C_j}{\lambda - \lambda_j} e^{2i(\lambda_j y + 4\lambda_j^3 t)} \begin{pmatrix} M_{12}(y, t, \lambda_j) \\ M_{22}(y, t, \lambda_j) \end{pmatrix}. \quad (4.2)$$

Evaluation at $\bar{\lambda}_n$, equation (4.2) becomes

$$\left(\frac{\overline{M_{22}(y, t, \lambda_n)}}{M_{12}(y, t, \lambda_n)} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^N \frac{C_j}{\bar{\lambda}_n - \lambda_j} e^{2i(\lambda_j y + 4\lambda_j^3 t)} \begin{pmatrix} M_{12}(y, t, \lambda_j) \\ M_{22}(y, t, \lambda_j) \end{pmatrix}, \quad n = 1, \dots, N. \quad (4.3)$$

Solving this algebraic system for $M_{12}(y, t, \lambda_j), M_{22}(y, t, \lambda_j), n = 1, \dots, N$, and substituting them into (4.1) provides a explicit expression for the $[M(y, t, \lambda)]_1$. In terms of the symmetries (3.17), we can get that $M_{12}(y, t, \lambda)$, which solves the Riemann Hilbert problem. Then

$$-i \lim_{\lambda \rightarrow \infty} (\lambda M(y, t, \lambda))_{12} = m(y, t) = -i \sum_{j=1}^N C_j e^{2i(\lambda_j y + 4\lambda_j^3 t)} M_{12}(y, t, \lambda_j).$$

Therefore, the N soliton solution $q(x, t)$ is expressed by the (3.7).

4.1 One-soliton solution

In this section we derive a explicit formulate for the one-soliton solution, which arise when $a(\lambda)$ has a pure imaginary λ_1 of simple zero. Letting $N = 1$ in (4.3), from the the symmetries of (2.10), we can deduce that $a(\lambda_1) = \overline{a(-\bar{\lambda}_1)} = 0$, then $\lambda_1 = -\bar{\lambda}_1$ and $\dot{a}(\lambda_1) = \overline{\dot{a}(-\bar{\lambda}_1)}$. Since the b_1 is a real constant, we find that $C_1 = -\overline{C_1}$, thus C_1 is a pure imaginary. Making use of the symmetries of (3.17), we can obtain

$$\begin{aligned} \overline{M_{22}(y, t, \lambda_1)} &= 1 + \frac{C_1}{\bar{\lambda}_1 - \lambda_1} e^{2i(\lambda_1 y + 4\lambda_1^3 t)} M_{12}(y, t, \lambda_1), \\ \overline{M_{12}(y, t, \lambda_1)} &= \frac{C_1}{\bar{\lambda}_1 - \lambda_1} e^{2i(\lambda_1 y + 4\lambda_1^3 t)} M_{22}(y, t, \lambda_1). \end{aligned}$$

Then,

$$\overline{M_{22}(y, t, \lambda_1)} = \frac{(\bar{\lambda}_1 - \lambda_1)^2}{(\bar{\lambda}_1 - \lambda_1)^2 + |C_1|^2 e^{2i(\lambda_1 y + 4\lambda_1^3 t)} e^{-2i(\bar{\lambda}_1 y + 4\bar{\lambda}_1^3 t)}}.$$

Substituting this result into the (4.3), we get

$$M_{12}(y, t, \lambda) = \frac{\overline{C_1}(\bar{\lambda}_1 - \lambda_1)^2}{(\lambda - \bar{\lambda}_1)[(\bar{\lambda}_1 - \lambda_1)^2 e^{2i(\bar{\lambda}_1 y + 4\bar{\lambda}_1^3 t)} + |C_1|^2 e^{2i(\lambda_1 y + 4\lambda_1^3 t)}]}. \quad (4.4)$$

Let $\lambda_1 = i\varepsilon$, $\varepsilon > 0$ and, In order to conveniently study the properties of the one soliton solution, we choose $C_1 = \pm 2i\varepsilon$. When $C_1 = -2i\varepsilon$, substituting both parameters into the (4.4), it comes into being

$$M_{12}(y, t, \lambda) = \frac{2i\varepsilon e^{-2(\varepsilon y - 4\varepsilon^3 t)}}{(\lambda + i\varepsilon)[1 - e^{-4(\varepsilon y - 4\varepsilon^3 t)}]}. \quad (4.5)$$

Then,

$$-i \lim_{\lambda \rightarrow \infty} (\lambda M(y, t, \lambda))_{12} = -(\arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)})_y.$$

where the $\arctan h x$ is the inverse function of $\tanh x$. Furthermore,

$$\begin{aligned} \int_y^\infty m(y', t) dy' &= -i \int_y^\infty \lim_{\lambda \rightarrow \infty} (\lambda M(y', t, \lambda))_{12} dy' \\ &= - \int_y^\infty (\arctan h e^{-2(\varepsilon y' - 4\varepsilon^3 t)})_{y'} dy' \\ &= \arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)}. \end{aligned} \quad (4.6)$$

The solution $q(x, t)$ in (3.7) can transforms into

$$q(x, t) = e^{8\arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)}} - 1. \quad (4.7)$$

Let $\alpha(y, t) = e^{\arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)}}$, we find that $\ln \alpha(y, t) = \arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)}$, then

$$\tanh(\ln \alpha(y, t)) = e^{-2(\varepsilon y - 4\varepsilon^3 t)}$$

i.e.

$$\frac{e^{\ln \alpha(y, t)} - e^{-\ln \alpha(y, t)}}{e^{\ln \alpha(y, t)} + e^{-\ln \alpha(y, t)}} = e^{-2(\varepsilon y - 4\varepsilon^3 t)},$$

we deduce

$$\alpha^2(y, t) = -\tanh^{-1}(-\varepsilon y + 4\varepsilon^3 t).$$

Equation (4.7) can be written as

$$q(x, t) = (e^{\arctan h e^{-2(\varepsilon y - 4\varepsilon^3 t)}})^8 - 1 = \tanh^{-4}(-\varepsilon y + 4\varepsilon^3 t) - 1. \quad (4.8)$$

Substituting y with x , (4.8) becomes

$$q(x, t) = \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x, t)) - 1 \quad (4.9)$$

where $\gamma(x, t) = \int_x^\infty (1 - \rho(\xi, t))d\xi$, $\rho(x, t) = \tanh^{-2}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x, t))$. Then (4.9) can be varied as $(1 + q(x, t))^{\frac{1}{2}} - 1 = \cosh^2(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x, t))$, hence the one soliton solution $q(x, t)$ has a singularity at the peak of the soliton so called cusp soliton.

When $\lambda_1 = i\varepsilon$ and $C_1 = 2i\varepsilon$, the corresponding one soliton solution $q(x, t)$ of (1.2) can be expressed

$$q(x, t) = \tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x, t)) - 1 \quad (4.10)$$

where $\gamma(x, t) = \int_x^\infty (1 - \rho(\xi, t))d\xi$, $\rho(x, t) = \tanh^{-2}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x, t))$.

4.2 Remark In this paper, we use the uniform method to obtain the solution $q(x, t)$ of equation (1.2) expressed by the (4.9) and (4.10). While the [3] applies the inverse scattering method to get the solution $q(x, t)$. If $\varepsilon = \kappa$ (κ in the [3]), To the one soliton solution, when $C_1 = -2i\varepsilon$, expression of the solution in both paper is similar, identically with $-\varepsilon x + 4\varepsilon^3 t$ in the

$$\tanh^{-4}(-\varepsilon x + 4\varepsilon^3 t - \varepsilon\gamma(x, t))$$

and $\kappa x - 4\kappa^3 t$ in the $\tanh^{-4}(\kappa x - 4\kappa^3 t - \kappa x_0 + \varepsilon_+)$ in [3]. There is different point that the expression of one soliton solution in the two papers, one is dependent of the $-\varepsilon\gamma(x, t)$ of x , the other is $-\kappa x_0 + \varepsilon_+$ of x .

Acknowledgments

This work was supported by grants from the National Science Foundation of China (Project No.11271079), Doctoral Programs Foundation of the Ministry of Education of China.

References

- [1] W. Hereman, P.P Banerjee and M R Chatterjee, Derivation and implicit solution of the Harry-Dym equation and its connections with the Korteweg-de Vries equation, *J. Phys. A:Mat Gen* **22**(1989), 241-255.
- [2] L. P. Kadanoff, Exact solutions for the Saffman-Taylor problem with surface tension, *Phys. Rev. Lett.* **65**, 2986-1990.
- [3] M. Wadati, Yoshi H, Ichikawa and Toru Shinizu, Cusp soliton of a new integrable nonlinear evolution equation, *Progress of theoretical physics* **64** (1980), 1959-1967.
- [4] F. Magri, A geometrical approach to the nonlinear solvable equations, *Nonlinear Evolution Equations and Dynamical Systems Lecture Notes in Physics* **120** (1980), 233-263.
- [5] M. Leo, R. A. Leo, G. Soliani, L. Solombrino, and L. Martina, Lie-Backlund symmetries for the Harry-Dym equation, *Phys. Rev. D* **26** (1980), 1406-1407.
- [6] C Rogers. M C Nucci, On reciprocal Backlund transformations and the Korteweg-deVries hierarchy, *Phys. Scr* **33** (1986), 289-292.
- [7] M. Leo, R. A. Leo, G. Soliani, L. Solombrino, On the isospectral-eigenvalue problem and the recursion operator of the Harry-Dym equation, *Phys. Scr* **38** (1983), 45-51.
- [8] P. P. Banerjeet, Faker Daoudt and Willy Hereman, A straightforward method for finding implicit solitary wave solutions of nonlinear evolution and wave equations, *J. Phys. A:Mat Gen J. Phys. A: Math. Gen.* **23** (1990) 521-536.
- [9] A. S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, *Proc. Roy. Soc. Lond. A* **453** (1997), 1411-1443.
- [10] A. S. Fokas, On the integrability of linear and nonlinear partial differential equations, *J. Math. Phys* **41** (2000), 4188-4237.
- [11] A. S. Fokas, A unified approach to boundary value problem, CBMS-NSF regional conference series in applied mathematics, SIAM (2008).
- [12] J. Lenells and A. S. Fokas, An integrable generalization of the nonlinear Schrödinger equation on the half-line and solitons, *Inverse Problems* **25** (2009), 115006 (32pp).

- [13] A. S. Fokas, J. Lenells, Explicit soliton asymptotics for the Korteweg-de Vries equation on the half-line, arXiv:0812.1579.
- [14] J. Lenells, An integrable generalization of the sineCGordon equation on the half-line, *IMA J. Appl. Math***76** (2011), 554-572.
- [15] J. Lenells and A. S. Fokas, On a novel integrable generalization of the nonlinear Schrödinger equation, *Nonlinearity* **22** (2009), 11–27.
- [16] A. Boutet de Monvel and D. Shepelsky, Riemann-Hilbert problem in the inverse scattering for the Camassa-Holm equation on the line, *Math. Sci. Res.Inst.Publ***55** (2008), 53–75.
- [17] A. Boutet de Monvel and D. Shepelsky, Inverse scattering transform for the Degasperis- Procesi equation: a Riemann-Hilbert approach, arXiv:1107.5995.